THE ERROR OF THE PLANE-LAYER APPROXIMATION AS APPLIED TO RADIATION IN THE BOUNDARY LAYER OF A GRAY GAS

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In the study of the high-temperature gas flow, radiation is represented in the energy equation by an integral over the temperature field. This makes the solution of specific problems quite difficult. For this reason various simplifications have been introduced. This, however, raises the question of the corresponding error. This error has been estimated for the case of a boundary layer in an optically-thin gas by Pavlova and Shmyglevskii [1].

In this paper I shall estimate the error associated with the plane-layer approximation as applied to the radiation in a boundary layer on a flat plate of length l in a parallel stream of an absorbing-emitting gray gas, with constant temperature outside the boundary layer and arbitrary, but constant, absorption coefficient. The problem is two-dimensional; local thermodynamic equilibrium is assumed; scattering is neglected; the wall is assumed to be black and its temperature is constant.

Under these assumptions, the radiation term in the energy equation is given in terms of the specific intensity of radiation φ by

$$-\operatorname{div}\mathbf{q} = \varkappa(c\varphi - 4\sigma T^4)\,. \qquad (1)$$

Here q is the radiation flux vector, T the gas temperature, σ the Stefan-Boltzmann constant, c the speed of light, and $\varkappa$ the absorption coefficient.

This relation follows from the equation of radiative transfer.

The specific intensity φ is given, under the present assumptions, by

$$c\varphi = \varkappa\frac{\sigma}{\pi}\iint\limits_{(x,\,y)} T^4(\xi,\ \eta)\,d\xi\,d\eta\int\limits_{-\infty}^{\infty} e^{-\varkappa r}\frac{d\zeta}{r^2} + \frac{\sigma}{\pi}T_0^4 y\iint\limits_{\Sigma} e^{-\varkappa r}\frac{d\sigma}{r^3} \qquad (2)$$

$$\left(r = \sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}\right).$$

The system of coordinates is chosen so that the x axis is directed along the plate, the y axis is normal to the plate, and the z axis is normal to the plane of flow. The double integration in the first term extends over the visible part of the plane, parallel to the xy plane and passing through the point x, y, z. In the second term the integration extends over the surface area of the plate Σ, with $d\sigma$ the area element of the plate surface and T_0 the plate temperature.

Equation (2) for φ can be transformed as follows:

$$c\varphi = \varkappa\frac{2\sigma}{\pi}\int\limits_0^{2\pi}\int\limits_0^{\rho_*} T^4 I_1(\varkappa\rho)\,d\rho\,d\theta + \frac{2\sigma}{\pi}T_0^4\int\limits_{\pi+\theta_2}^{2\pi-\theta_1} I_2(\varkappa\rho_*)\,d\theta \qquad (3)$$

where

$$\xi - x = \rho\cos\theta, \qquad \eta - y = \rho\sin\theta$$

$$I_n(t) = \int\limits_1^{\infty} e^{-t\alpha}\frac{d\alpha}{\alpha^n\sqrt{\alpha^2 - 1}} \quad (n = 1,\,2), \qquad \rho_* = \begin{cases} -y\csc\theta & \text{for } \pi + \theta_2 < \theta < 2\pi - \theta_1 \\ \infty & \text{for } -\theta_1 < \theta < \pi + \theta_2\,. \end{cases}$$

If the temperature in (2) or (3) is a function of η only, and for $\eta > 0$ is the same as at $\xi = x$, while for $\eta \le 0$ it is constant and equal to the plate temperature, then the specific intensity of radiation and, consequently, div q can be expressed in terms of E_n functions ($n = 1, 2, 3$). This is the plane-layer approximation that we are considering.

The difference between the exact and the approximate expression for the specific intensity is

$$\Delta c\varphi = \varkappa\frac{2\sigma}{\pi}\int\limits_0^{2\pi}\int\limits_0^{\rho_*}[T^4(\xi,\ \eta) - T^4(x,\ \eta)]\,I_1(\varkappa\rho)\,d\rho\,d\theta \qquad (4)$$

This is the variable the magnitude of which we want to estimate. Let the plane of flow be divided into several regions (Fig. 1). The value of δ is chosen in such a way that the temperature in Region 1 is constant, but then $\delta \ll l$, so that

$$\cos\theta_1' \approx \cos\theta_1'' \approx 1\,.$$

The point under consideration lies in Region 3, sufficiently far from the edges of the plate.

In accordance with our choice of δ, the integrand of (4) vanishes in Region 1.

An estimate of the integral over Region 2 yields

$$|\Delta c\varphi_2| \leqslant \sigma (T_2^4 - T_1^4) \frac{2}{\pi} \int_1^\infty \frac{d\alpha}{\alpha \sqrt{\alpha^2 - 1}} \left[\int_0^{\theta_1''} d\theta \int_{\vartheta_1}^{\vartheta_0} e^{-t}\, dt + \int_0^{\theta_1'} d\theta \int_{\vartheta_1}^{\vartheta_2} e^{-t}\, dt \right] \leqslant$$

$$\leqslant \sigma (T_2^4 - T_1^4) \frac{\delta}{l} e^{-\varkappa l} [1 - E_2(\varkappa l)] \qquad \left(\vartheta_0 = \varkappa \frac{\delta - y}{\sin\theta},\ \vartheta_1 = \varkappa \frac{l}{\cos\theta},\ \vartheta_2 = \varkappa \frac{y}{\sin\theta} \right).$$

Here T_2 and T_1 are the maximum and minimum values of the temperature, respectively. The estimate for Region 4 leads to the same result.

To obtain an estimate for Region 3 we use two terms of the series

$$T^4(\xi, \eta) - T^4(x, \eta) = \sum_{n=1}^{\infty} \left. \frac{\partial^n T^4}{\partial \xi^n} \right|_{\xi = x} \frac{(\xi - x)^n}{n!} = \sum_{n=1}^{\infty} \frac{\partial^n T^4}{\partial x^n} \frac{\rho^n}{n!} \cos^n \theta,$$

as the first term (and all other odd terms) vanishes. Thus

$$|\Delta c\varphi_3| \leqslant 2\varkappa\sigma \left\langle \left| \frac{\partial^2 T^4}{\partial x^2} \right| \right\rangle \frac{2}{\pi} \int_1^\infty \frac{d\alpha}{\alpha \sqrt{\alpha^2 - 1}} \left[\int_0^{\theta_0} \cos^2\theta\, d\theta \int_0^{l \sec\theta} \rho^2 e^{-\varkappa\rho} d\rho + \right.$$

$$\left. + \int_{\theta_0}^{\frac{1}{2}\pi} \cos^2\theta\, d\theta \int_0^{\delta \cos\theta} \rho^2 e^{-\varkappa\rho} d\rho \right] \approx 2\sigma \left\langle \left| \frac{\partial^2 T^4}{\partial x^2} \right| \right\rangle \varkappa \left[\frac{\delta}{l} \int_0^l \rho^2 e^{-\varkappa\rho} d\rho + \int_0^\delta \beta^2 d\beta \int_1^{l/\delta} t e^{-\varkappa\beta t} dt \right] \approx$$

$$\approx 2\sigma \left\langle \left| \frac{\partial^2 T^4}{\partial x^2} \right| \right\rangle \frac{1}{\varkappa^2} \frac{\delta}{l} \left\{ -\varkappa l (\varkappa l + 1) e^{-\varkappa l} + \frac{l}{\delta} [2 - (\varkappa\delta + 2) e^{-\varkappa\delta}] \right\}.$$

Here we used the approximation $\tan\theta_0 = \delta/l$, $\cos\theta_0 \approx 1$. The angular brackets represent mean values. The second derivative $\partial^2 T^4/\partial x^2$ is of the order of magnitude of $(T_2^4 - T_1^4)/l^2$, which is used in the final results.

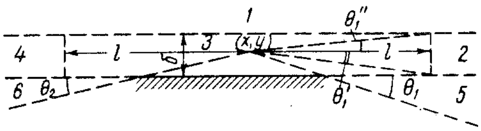

Fig. 1.

In estimating the right side of (4) for Region 5 we assume that the point under consideration is sufficiently far from the edges of the plate, which leads to the inequality

$$|\Delta c\varphi_5| \leqslant \frac{2}{\pi} \sigma (T_2^4 - T_1^4) \int_1^\infty \frac{d\alpha}{\alpha^2 \sqrt{\alpha^2 - 1}} \int_0^{\theta_1} \exp \frac{(-\varkappa y) d\theta}{\sin\theta} \leqslant \frac{2}{\pi} \sigma (T_2^4 - T_1^4) \frac{\delta}{l} E_2(\varkappa l).$$

The same estimate is obtained for Region 6. The sum of all these estimates is

$$|\Delta c\varphi| \leqslant 2\sigma (T_2^4 - T_1^4) \frac{\delta}{l} \left\{ e^{-\varkappa l} [1 - E_2(\varkappa l)] - \frac{\varkappa l + 1}{\varkappa l} e^{-\varkappa l} + \right.$$

$$\left. + \frac{2 - (\varkappa\delta + 2) e^{-\varkappa\delta}}{\varkappa^2 l \delta} + \frac{2}{\pi} E_2(\varkappa l) \right\}.$$

The error in div q is related to the error in the specific intensity by the equation

$$|\Delta\, \mathrm{div}\, \mathbf{q}| = \varkappa\, |\Delta c\varphi|,$$

which follows from (1).

If we now require that the error in div q due to the plane-layer approximation shall not exceed in order of magnitude the error in the energy equation admissible in boundary-layer theory, then the condition for the applicability of the plane-layer approximation for this case is

$$\varkappa l\left\{e^{-\varkappa l}[1-E_2(\varkappa l)]+\frac{2}{\pi}E_2(\varkappa l)\right\}-(\varkappa l+1)\,e^{-\varkappa l}+\frac{2-(\varkappa\delta+2)\,e^{-\varkappa l}}{\varkappa\delta}\leqslant$$

$$\leqslant\frac{\rho_* u_* c_{p_*}(T_2-T_1)}{2\sigma(T_2^4-T_1^4)}\,\frac{\delta}{l} \qquad (5)$$

Here the variables with the star subscript represent characteristic values of the parameters. Figure 2 represents (5) with the equality sign and $\delta/l = 0.01$.

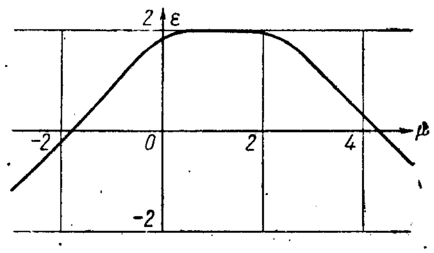

Fig. 2.

The abscissas represent $\mu = \log \varkappa l$. The ordinates represent

$$\varepsilon=\lg\frac{\rho_* u_* c_{p_*}(T_2-T_1)}{2\sigma(T_2^4-T_1^4)}\ .$$

The region of applicability of the plane-layer approximation lies above the curve. The curve has the asymptotes

$$\varepsilon=-\mu+\lg[2(l/\delta)^2] \quad \text{for} \quad \mu\to+\infty$$

and

$$\varepsilon=\mu+\lg(2l/\pi\delta) \quad \text{for} \quad \mu\to-\infty$$

One can see that there exists a value of ε such that for all higher values the plane-layer approximation for boundary-layer radiation is valid for all values of $\varkappa l$. For $\delta/l > 0.01$ the curve lies lower and the region of its maximum is narrower.

It should be noted that in practical problems the region occupied by the emitting-absorbing gas is finite, and the estimate derived here may be inapplicable in the optically-thin case.

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REFERENCES

1. L. M. Pavlova and Yu. D. Shmyglevskii, "The boundary layer in a radiating gas," PMTF, no. 1, 1964.

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